An Initial Value Problem for the Horizontal Infiltration of Water

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An initial value problem for the horizontal infiltration of water is solved by means of a perturbation technique. The diffusivity is assumed to vary as a positive power of the normalized water content. Evolution of the profile is supposed to be taking place in two stages. In the first stage, moisture content at the origin increases steadily as some arbitrary power of dimensionless time until the maximum normalized value of unity is attained; thereafter, in the second stage, moisture content at the origin remains at the constant value of unity. The moisture distribution at the end of the first stage defines the initial values of moisture for the problem in the second stage. This paper derives the flow details for the second stage, and constructs moisture profiles as they evolve for times greater than unity.

INTRODUCTION

Problems of moisture movement in soils are generally described by nonlinear partial differential equations. For moist soils, the diffusivity and conductivity are such that their values at lower water contents become practically negligible when compared to the values at higher water contents. The well-known cases of exponential diffusivity functions [Meador et al., 1972], soils whose matrix flux potentials are exponentials in pressure heads [Neer, 1972, p. 492], and others [Swartzendruber, 1969, p. 220] are examples of this behavior. Such rapidly changing coefficients force the differential equation to become "singular" at these lower water contents, generating the often observed sharp wetting "fronts" that terminate abruptly the moisture profiles. The strong nonlinearities and singularities combine to make the analysis of the phenomena difficult.

Therefore a fairly large body of numerical schemes of solution has evolved in the literature. Finite difference schemes [Hanks and Bowers, 1962; Haoerzap et al., 1977], and finite element methods [Neuhaus, 1971; Bruch, 1975] presently constitute two of the standard techniques of solution for these problems.

Solutions based upon analytic and quasianalytic approaches are relatively few in number. Constant moisture conditions, or constant flux conditions, at the boundaries are frequently the starting point in all these studies [Philip, 1969; Philip and Knight, 1974; Brusseau, 1974], referring to an earlier work by Headley and Alskie [1961], constructed solutions by specifically referring to the heavy nonlinearity of the coefficients. Utilizing an iterative procedure, Farlange [1971] derived formulae and expressions in a general framework. These iterations improve considerably in convergence if Ciferri's [1974] corrections are adopted. Hanks [1976a,b] derived solutions based upon perturbation techniques that explicitly take advantage of the high nonlinearities in the equations.

THE EQUATIONS

A problem of horizontal infiltration of water into an initially dry, seminfinite homogeneous soil column is analyzed in the following sections. The nonlinear diffusion equation

\[
\frac{\partial}{\partial x} \left( D(x) \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial t}
\]

is specialized to

\[
F(0) = 0 \quad n > 0
\]

and is subject to the following conditions

\[
\theta = \frac{r_0}{r} \quad 0 \leq r \leq 1 \quad x = 0 \quad m > 0
\]

\[
\theta = \frac{r_0}{r} \quad 1 \geq r \geq 1 \quad x = 0 \quad m > 0
\]

\[
\theta = 0 \quad \theta_0 \frac{\partial}{\partial x} = 0 \quad x \geq R(t)
\]

\[
\frac{d}{dt} \int_0^x \theta(x, t) \, dx = \int_0^x \frac{\partial}{\partial t} \, dx
\]

All the variables used here are in dimensionless form, and are related to the physical variables with asterisks

\[
\theta = (1 - f_{\text{m}}(x)) \left( f_{\text{m}}(x) - f_{\text{m}}* \right)
\]

\[
D(x) = D^*(x) / D^*(f_{\text{m}}*)
\]

\[
x = x^* \theta(x) / D^*(f_{\text{m}}*)
\]

\[
r = r^* \theta(x) / D^*(f_{\text{m}}*)
\]

\[D^* \text{ is diffusivity and } A^* \text{ is conductivity of soil.}
\]

Values of \( A^* \) are not needed if a characteristic time or length is available. The water content \( \theta \), and the diffusivity \( D^* \), are both normalized to 1. \( x \) is the distance and \( t \) the time in dimensionless form. The function \( R(t) \) denotes the distance of the wetting front; it is unknown to begin with, and will be determined as part of the solution.

Condition (1b) means that \( \theta \) increases monotonically at the supply surface \( x = 0 \) until \( t = 1 \); thereafter, this moisture content remains constant for all \( t \geq 1 \). Condition (1c) states that, at and beyond \( R(t) \), the moisture and flux both vanish identically. In other words, the \( \theta(x, t) \) profile hits the \( x \) axis and terminates abruptly at \( R(t) \), thus forming a front there. Beyond this front, no disturbances are propagated into the medium.
Note that it has been assumed in the above that
\[ D^*(p) = D^*(p_{\infty}) + \left( p - p_{\infty} \right) \frac{\partial D^*(p)}{\partial p_{\infty}} \cdot \overline{p_{\infty}} - \overline{p_{\infty}}^2 \]

Since \( D^*(p_{\infty}) = 0 \), it may be inferred that the following analysis holds for those soils that do not transport moisture beneath a certain concentration of \( p_{\infty} \). This \( p_{\infty} \) may be identified with the residual moisture content of dry soils. The above relation does not imply that for all soils \( D^*(p) = 0 \). However, the very rapid variation of diffusivity function for soils in general, is portrayed reasonably well by this power law.

Relation (1d) is very important for the present analysis. It is derived by an integration by parts of (1e), utilizing (1b) and (1c). Babu [1978a, b, c, 1977] employed this relation extensively in his papers for the special case of constant water content \( f(t) = 1 \). From (1d) follows a first estimate for the distance of the wetting front at \( t \) during \( 0 \leq t \leq 1 \).

\[
\int_{0}^{1} \frac{\partial N}{\partial t} \, dx = \int_{0}^{1} \frac{1}{dx} \, dx = \int_{0}^{1} \frac{\partial D^*(p)}{\partial p_{\infty}} \left( \overline{p_{\infty}} - \overline{p_{\infty}}^2 \right) \left( x + 1 \right) \left( m(x) + 1 \right) + 1 \]  
(1e) Introducing the parameter \( \epsilon \) (usually small), \( \epsilon = 1/(n + 1) \), leads to the estimates
\[ f(t) = O(2/n^{1/2}/m(x + 1)^{1/2}) \quad \text{for} \quad 0 < t < 1 \]  
(1f)

Here \( O \) is the big-order symbol. Since \( \epsilon \) is usually small in applications, and since the profile lengths are estimated in terms of \( m \), it is natural to identify the penetration parameter with \( m \). From later sections, it will be seen that relation (1f) plays an important role in the construction of solutions.

A particular case of (1a) deserves mention here. If \( m = 1 \) in (1a) the set of equations (1a)-(1d) describes flow patterns associated with the well-known theories on groundwater by Dupuit and Forchheimer (Bear, 1972; Kirkham and Powers, 1972). A slight modification of coefficients in (1a) would describe some highly important cases of axisymmetric flows associated with pumping of aquifers.

THE PROBLEM

The solution to this problem is constructed in two steps. The first step relates to the period \( 0 \leq t \leq 1 \), wherein the moisture content at the supply point increases. In the second step, the moisture content at the origin remains constant for all \( t \geq 1 \). Using a perturbation technique, Babu and van Genuchten (1979) constructed the first step solution. In this paper, the flow details for the second stage \( 1 \leq t \leq \infty \) will be described. In effect, then, the problem to be solved here is an initial value problem: the initial values of \( \theta \) for the second step problem are the final values of the solution in the first step.

A PREVIOUS RESULT: THE FIRST STEP SOLUTION

For the period \( 0 \leq t \leq 1 \), it is possible to reduce (1a) to an ordinary differential equation by employing similarity transformations. The relevant transformations and the solution are given below. The zero subscript refers to this first stage.

\[ \frac{\partial \phi}{\partial x} = 2 + \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} + \ldots \]  
(2)

Assuming the solution in the form
\[ \theta = c^{1/2} \phi(x, m) \]  
(3)

the following differential equation is obtained
\[ \frac{\partial^2 \phi}{\partial x^2} = \phi(x, m) \quad \frac{x^{1/2} = \frac{1}{2} \left( 1 - 1 \right)}{\partial x} \]  
(4)

and the conditions
\[ \phi = 0 \quad \theta = 1 \]  
(5)

\[ \phi = 1 \quad \theta = 0 \]  
(6)

The function \( \phi(x, m) \) is associated with the unknown location of the wetting front, and is determined by successive approximations. A direct use of (1d) gives

\[ \int_{0}^{1} V^2 N_{m} \, dx \phi(x, m, \left( 1 + \frac{1}{m(x + 1)^{1/2}} \right) = 1 \]  
(7)

The first few terms in the expansion of \( \phi \) are given in (2) for reference. The solution to (1a) is satisfied valid for all \( x \) and \( t \leq 1 \), and which remains free from singularities, is determined as

\[ \theta = c^{1/2} \phi(x, m, \epsilon) \quad \frac{x^{1/2} = \frac{1}{2} \left( 1 - 1 \right)}{\partial x} \]  
(8)

where

\[ V_{m} = \frac{1}{2} + \frac{1}{2} \left( 1 + \frac{1}{m(x + 1)^{1/2}} \right) \]  
(9)

The location of the front is given by

\[ \theta = 1 \quad \frac{x^{1/2} = \frac{1}{2} \left( 1 - 1 \right)}{\partial x} \]  
(10)

THE SECOND STAGE PROBLEM

For \( t \geq 1 \), the solution does not exhibit similarity characteristics. Only for asymptotically large \( t \) will the profiles again exhibit the well-known \( m^{1/2} \) behavior. The problem to be solved in the second stage is thus

\[ \theta = c^{1/2} \phi(x, m, \epsilon) \quad \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x} + \left( 1 - 1 \right) - \frac{\partial \phi}{\partial x} \]  
(11)

with the conditions

\[ x = 0 \quad \theta = 1 \]  
(12a)

\[ x = \theta \quad \frac{\partial \phi}{\partial x} = 0 \]  
(12b)
Scaling

First, a correct scale for the independent variables \(x\) and \(t\) will be chosen. These variables will become unbounded eventually. An integration of (14) yields for \(t \geq 1\)

\[
\int_{t}^{x} \Delta \tau(x, t) \, dx = \int_{x}^{t} \Delta \tau(x, t) \, dx = \int_{x}^{t} \frac{d\tau}{\Delta \tau(x, t)} = -\log(1-\frac{x}{t})
\]

Use of (14) with \(t = 1\) gives a basic estimate through (14).

\(t \geq 1\)

\[
\int_{t}^{x} \Delta \tau(x, t) \, dx = -(1-\frac{x}{t}) + \frac{x}{t}(1+\frac{1}{2} + \cdots)
\]

Equation (15) thus leads to a suitable scale for \(x\)

\[
\sigma = \frac{x}{\Delta \tau(x, t)}
\]

where

\[
\Delta \tau(x, t, m) = \frac{2}{(1+\psi_0 + \cdots)} \left( \frac{m}{m(1+\psi_0 + \cdots)} \right)
\]

To match conditions at \(x = 1 \pm 0\), it is necessary that

\[
\phi(x, 1, m) = \phi(x, m)
\]

Relation (17) ensures that \(\sigma = \eta\) at \(t = 1\).

The differential equation (12) now transforms into \(i = i(x, t, \tau)\)

\[
\frac{\partial \psi}{\partial \tau} = \psi \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x \partial \tau} \right)
\]

It is also necessary to preserve the time derivatives in the equations, so that the time dependency of the solution is, as far as is possible, accurately pictured. For this purpose, and also to achieve some simplification of the work, a final scaling on \(\tau\) is introduced:

\[
\exp(1-\tau) = 1 + (1-\eta)m + k\eta + \cdots
\]

or

\[
\tau = \log(1 + (1-\eta)m + k\eta + \cdots)
\]

It is seen from (19) that the new time variable \(\tau\), while large, grows much slower than the original time \(t = 1\). Such transformations generally prove quite useful in analyzing the large time behavior of the flow patterns.

The Equations

In terms of the new time variable \(\tau\), the problem finally reduces to

\[
\frac{\partial^2 \psi}{\partial \tau^2} = \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x^2} \right) - \psi \frac{\partial^2 \psi}{\partial x \partial \tau} - \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x \partial \tau} \right)
\]

where the overdot denotes differentiation on \(\tau\). The initial and boundary conditions associated with (20) are

\[
\tau = 0 \quad \psi = \eta \quad \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \psi \quad \eta = 0
\]

\(\tau = 1 \quad \psi = 0 \quad \frac{\partial \psi}{\partial \tau} = 0 \quad \eta = 1\)

and

\[
\eta = 1 \quad \psi = \eta \quad \frac{\partial \psi}{\partial \tau} = 0 \quad \eta = 0, \quad \frac{\partial \psi}{\partial \tau} = 0
\]

The Perturbation Scheme

The solution is assumed to resemble (9) of stage one.

\[
i = \psi(x, t, m) = \psi(1-\psi_0 + \cdots)
\]

where the functions \(\psi_0, \psi_1, \cdots\) are to be determined so as to satisfy conditions (21)-(24). The technique of solution is to expand (25) in powers of \(\tau\), substitute this into (20), and successively equate to zero the coefficients of like powers of \(\tau\).

From (25), it follows that

\[
i = 1 + \eta(x, t, m) + \eta(1-\eta) + \eta^2 + \cdots
\]

and

\[
\eta = \log(1 - \eta) + \frac{1}{2} \log(1 - \eta) + \cdots
\]

Substitution of (25a) and (26b) into (20-24) results in the following sequence of problems. It should be pointed out here that the following equations, when simplified, will automatically be freed of the singular terms like \(1/(1-\eta)\), \(1/(1-\eta^2)\), etc. It was specifically for this purpose that the factor \((1-\eta)^{-1} + \cdots\) had to be introduced into (9) and (25).

\(\eta\) terms

\[
\frac{\partial \eta}{\partial \tau} = 0
\]

\(\eta\) terms

\[
\frac{\partial \psi}{\partial \tau} = 0
\]

This reduces, after cancellation of the singularities, to

\[
\frac{\partial \psi}{\partial \tau} = 0
\]

\[
\eta = 1 \quad |\eta| < \varepsilon \quad \text{(boundedness)}
\]
In this paper, only terms in \( t \) and \( \eta \) are computed. No attempt has been made to evaluate the higher-order approximations \( f_\infty \), \( f_\infty \ldots \).

**The Solution**

First approximation in \( t \) terms: Relation (27e) directly yields

\[
\phi_t = \frac{3}{2} \quad \phi_\infty = 0
\]

The solution of (27a) is taken in the form

\[
i_t = -\frac{3}{2} + \sum X_i(t) \phi_i \left[ (1 - \eta)\frac{\partial}{\partial \eta} \right] (1 - \eta)^{\nu - 1}
\]

leading to

\[
\frac{dA_\infty}{dt} = \frac{\mu_\infty^2}{8} A_\infty = 0
\]

Here \( J_\nu \) is the Bessel function of order one, and \( \mu_\infty > 0 \) are the successive zeros of this function: \( J_\nu(\mu_\infty) = 0 \). Certain elementary properties of the \( J_\nu(x) \) (Bray, 1968, sect. 70), along with conditions (26b), (26d) give the first order solution

\[
\hat{I}(x, t) = \frac{3}{2} - 16 \sum R_i(t) \frac{\phi_i \left[ (1 - \eta)\frac{\partial}{\partial \eta} \right]}{\phi_i (1 - \eta)^{\nu - 1}}
\]

where the exponential factor \( E(\tau, t) \) is given by

\[
E(\tau, t) = \exp \left( -\mu_\infty (t/\delta) \frac{1}{\mu_\infty^2} \right)
\]

and where use has been made of the following Fourier-Bessel expansion:

\[
\eta = -16 \sum J_\nu(\lambda_\nu) X_i (1 - \eta)^{\nu - 1}
\]

(\( \phi_i \) is the zero-order Bessel function of first kind).

Second approximation in \( t \) terms: Computation of \( d\phi_t \) is achieved by using (31a) in (28e). The sum

\[
\sum_{1 = 1}^{1} \frac{1}{12} \frac{1}{12} = \frac{1}{12}
\]

[Tanner, 1968, p. 70], along with simple integrations involving (31c), leads to the result

\[
\phi_t = \frac{3}{2} - 64 \sum E_i(\tau) \frac{1}{\mu_\infty^2}
\]
Next, from (29a), the equation to be solved is taken as
\[
\frac{\partial^2 \theta}{\partial x^2}(x, t) - \frac{\partial \theta}{\partial t} = (1 - \eta) \frac{\partial \theta}{\partial t} - 2 \frac{\partial^2 \theta}{\partial x^2} + (2 - \eta) \frac{\partial \theta}{\partial t} + \left(1 - \frac{\eta}{2}\right) \sum E(x) \quad \text{(33)}
\]
\[
\eta = \frac{1}{12} \sum E(x)
\]
If solved as before (for \(\hat{i}_r\)), the resulting expression for \(i_d(r, t)\) from (33), although exact, will be complicated enough to warrant some approximation of the right-hand side of (33). An expansion in powers of \((1 - \eta)\) leads to the following approximate relations:
\[
i_d \sim -\sum E(x) \quad \text{(34)}
\]
\[
\frac{\partial \theta}{\partial x} = \frac{1}{2} + \sum E(x) \quad \text{(35)}
\]
With the approximations (34), (33) takes on a simplified form
\[
\frac{\partial^2 \theta}{\partial x^2}(x, t) - \frac{\partial \theta}{\partial t} = \left(1 - \frac{\eta}{2}\right) \sum E(x) \quad \text{(35a)}
\]
\[
\text{The initial and the boundary conditions are}
\]
\[
r = 0 \quad i_d = \left(1 - \frac{1}{12}\right) h - \frac{\eta^2}{12} \quad \text{(35b)}
\]
\[
\eta = 0 \quad i_d = 0 \quad \text{(35c)}
\]
\[
\eta = 1 \quad |i_d| < \infty \quad \text{(35d)}
\]
The solution again emerges as
\[
i_d = \left(1 - \frac{1}{12}\right) h - \frac{\eta^2}{12} + \sum B_0(\theta)(\frac{\partial \theta}{\partial t} - \frac{\eta}{12}) \text{ \quad (where)} \quad \text{(36)}
\]
where the Fourier-Bessel coefficients \(B_0(\theta)\) are given by
\[
B_0(\theta) = -\left(A_4 + 2i_B + C\theta / \mu_d \mu_u \right) \text{ \quad and} \quad A_4 = \frac{1}{\mu_d} \exp(-\mu_d \sqrt{\mu_u} / \mu) \left(1 + \frac{1}{m} \right) \frac{\eta}{\mu_d} \quad \text{(37)}
\]
\[
\text{with } B_0(\theta) \text{ as defined in (37).}
\]
\text{Comment. Apparently, the occurrence of the factor } r \text{ in the middle term } (2r) \text{ of (37) tends to increase the values of } B_0(\theta) \text{ for large } r. \text{ Higher-order approximations will generally produce higher powers of } r. \text{ However, the exponential factor } \exp(-\mu_d \sqrt{\mu_u} / \mu) \text{ damps out much of this tendency. The question of whether this kind of influence could be neutralized by additional scaling devices will not be taken up here.}

\text{The Results}

To summarize, the solution computed up to second order approximations, is given by the following:
\[
\begin{align*}
\mu_1 &= 3.832 & \mu_2 &= 7.016 & \mu_3 &= 10.173, \cdots \\
E(x) &= \frac{1}{\mu_1 (2\mu_1)} [1 + (r - 1)(r - \mu_1 / \mu)] & \mu &= \frac{\exp(-\mu_d \sqrt{\mu_u} / \mu)}{\mu_d (1 - \eta)^{1/2}}
\end{align*}
\]
\[
r = \frac{1}{\mu_1} \ln[1 + (r - 1)(r + \mu_1 / \mu)] & \quad r \geq 1 \quad r \geq 0
\]
\[
\theta(x, r, m) = \frac{2m}{\mu + r} \left[1 + (r - 1)(r + \mu_1 / \mu)^{1/2} \right] \quad \text{(39)}
\]
\[
\eta = \frac{1}{r} \exp(-\mu_d \sqrt{\mu_u} / \mu) \left(1 + \frac{\eta}{2} \right) \frac{1}{\mu_1 (1 - \eta)^{1/2}} \quad \text{(40)}
\]

The moisture content \(\theta\) is given by
\[
\theta(x, r) = \left(1 - \eta^2\right) \exp(-\mu_d \sqrt{\mu_u} / \mu) (1 + \partial_x + \partial^2_x + \cdots y) \quad \text{(41)}
\]
The wet front location is given by (for any time \( t > 1 \))

\[
x(\text{front}) = L(t) = c_{\text{diff}}(t, x, m/m)\lambda
\]

\[(44)\]

**Comparison**

The solution given by (37)-(44) was compared with some numerical solutions in Figures 1 and 2. The numerical solution was constructed by a finite element method, utilizing Hermite polynomials as the basis functions [van Genuchten, 1978b]. It is obvious that the two solutions agree very well. In Figure 1, for \( m = 0.5, x = 2.0, \) and \( z = 5.0, \) the successive approximations by the perturbation techniques are drawn against the finite element solution. In Figure 2, for \( m = 2.0, x = 2.5, \) the second approximation solutions alone are shown against the finite element solution, at times \( r = 1.0, 2.0, \) and 3.0. It should be recalled that in this analysis, the initial value of time is \( r = 1.0. \)

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