
Kun Tu1,2,3, Qiang Wu1,2,4, Jirka Simunek3, Chaofan Chen4,5, Ke Zhu1,2, Yifan Zeng1,2, Shengheng Xu1,2, and Yang Wang6

1College of Geoscience and Surveying Engineering, China University of Mining and Technology, Beijing, China, 2National Engineering Research Center of Coal Mine Water Hazard Controlling, Beijing, China, 3Department of Environmental Sciences, University of California, Riverside, CA, USA, 4Helmholtz Centre for Environmental Research (UFZ), Leipzig, Germany, 5Applied Environmental Systems Analysis, Dresden University of Technology, Dresden, Germany, 6Central Plains Institute of Intelligent Geology, Zhengzhou, China

Abstract In this study, a general analytical model for the single-well circulation system is developed to analyze transient drawdown in a confined aquifer. The analytical solution of transient drawdown in the Laplace domain, which is numerically inverted into the time domain using the Stehfest method, is derived by employing a combination of the Laplace and Fourier cosine transforms. The characteristics of transient drawdown and the effects of different parameters related to the single-well circulation system on drawdown are investigated. Furthermore, the analytical solution under steady-state conditions is obtained using the Fourier cosine transform. The results show that steady drawdown contours are symmetric around a horizontal midplane of an aquifer and vary tremendously with distance from the well axis. The contours of drawdown around the sealed section are dense, meaning that the hydraulic gradient in this area is relatively large. The sensitivity analysis, performed to evaluate the characteristics of drawdown to changes in each parameter, indicates that the radial hydraulic conductivity and the length of the sealed section have a large impact on the drawdown and that each parameter has its influence period on the drawdown.

1. Introduction

Among different types of groundwater heat pump systems, the single-well circulation groundwater heat pump system is different from conventional groundwater heat pump systems—for example, a well doublet system (Banks, 2009; Galgaro & Cultrera, 2013) or a standing column well system (Abu-Nada et al., 2008; Deng et al., 2005)—especially in terms of the well structure. In a single-well circulation system (Figure 1), a single borehole is divided into two sections by well packers (the gray section in Figures 1 and 2), which are placed in the middle of a single borehole to block injected water in the injection section of the well from flowing to the pumping section of the well. The single-well circulation systems can be regarded as a combination of two partially penetrating wells. The lower one is a pumping well, from which groundwater is pumped at a rate Q, and the upper one is an injection well for injecting water at the same rate Q.

A single-well circulation groundwater heat pump system is a new technology that emerged in 2001 and since then has been employed to provide heating and cooling for buildings in China (Xu & Rybach, 2003). This unique groundwater heat pump system has gained tremendous interest and has been increasingly used during the last two decades, especially in northern China. Although the system has many practical applications, the theoretical investigation of this system is somewhat limited. Xu and Rybach (2005) presented the development and several successful engineering implementations of this original and innovative technology in terms of system designs, energy savings, and environmental protection. Wu et al. (2015) compared the single-well circulation system with other conventional groundwater heat pump systems in terms of the well configuration, requirements on hydrogeological and thermal geological conditions, environmental impacts, regulatory requirements, and so forth. Rybach (2015) presented engineering applications in China and other countries and summarized operational principles of the single-well circulation groundwater heat pump...
system. Finally, Zeng et al. (2017) provided a detailed discussion on energy efficiency, cost-effectiveness, operation, and maintenance of this system using a case study in Nebraska.

The heat exchange in a single-well circulation groundwater heat pump system is a result of combined groundwater flow and heat transport. Heat convection, in particular, is influenced by fluid flow. Therefore, a good understanding of groundwater flow is an especially vital and essential step before one can start studying the heat exchange in a single-well circulation groundwater heat pump system. Up to now, most of the research on groundwater flow focused on fully or partially penetrating wells.

The topic of drawdown around a fully penetrating well has been discussed in many works. For instance, Theis (1935) was the first who presented an analytical solution describing groundwater flow toward a fully penetrating well in a confined aquifer of an infinite extent. Later, Papadopoulos and Cooper (1967) proposed a mathematical model describing groundwater flow in a confined aquifer while considering the influence of the well radius and wellbore storage. Chen (1984) derived, based on several existing solutions, an analytical solution describing groundwater flow in a finite confined aquifer with a zero drawdown condition at the exterior boundary. Novakowski (1989) obtained a Laplace domain solution of drawdown in a composite confined aquifer while considering the effects of the well radius and wellbore storage. Subsequently, Yeh et al. (2003) presented an analytical solution describing the head distribution for a slug test in a two-zone confined aquifer system. Wen and Zhan (2008)
proposed a general analytical model describing non-Darcian groundwater flow in a confined aquifer by employing a combination of the power law function and a linearization procedure. Wang et al. (2012) presented an analytical solution describing groundwater flow while considering a constant-flux pumping along a finite radius well in the skin zone. Lin et al. (2016) proposed an analytical model describing groundwater flow in a composite confined aquifer of a finite extent induced by a pumping test while considering the effect of the well skin and a Robin-type outer boundary.

There are also a lot of studies available in the literature regarding groundwater flow to a partially penetrating well (Table 1). The first mathematical model for a partially penetrating well was presented by Hantush (1957), who derived an analytical solution by applying the Laplace and Fourier cosine transforms. Since then, partially penetrating well systems have been investigated by many other researchers (e.g., Ataie-Ashtiani et al., 2012; Chang & Chen, 2002; Chang & Yeh, 2009; Chen et al., 2010; Chiu et al., 2007; Yang & Yeh, 2005; Zlotnik et al., 1998). For instance, steady-state flow in an unconfined aquifer to one or multiple partially penetrating wells was studied by Luther and Haitjema (1999). Yang and Yeh (2005) presented a semianalytical solution describing groundwater flow toward a partially penetrating well induced by the constant head test in an infinite extent aquifer. Taking into account the effect of a finite-thickness skin, Chiu et al. (2007) presented an analytical solution for a drawdown of groundwater flow to a partially penetrating well. Barua and Bora (2010) presented a steady-state solution for groundwater flow induced by pumping in a partially penetrating well in a finite confined aquifer associated with the condition of the constant head at the outer boundary. Ataie-Ashtiani et al. (2012) derived an analytical solution for the capture zone of a partially penetrating well with an infinitesimal small well radius and a fixed pumping rate \( Q \). By employing the Ibash equation and a linearization procedure, Wen et al. (2013) presented an approximate analytical solution, which can describe the non-Darcian flow toward a partially penetrating well in a confined aquifer. Feng and Wen (2016) proposed an analytical model describing non-Darcian flow toward a partially penetrating well in a confined aquifer, which considered the effects of the well skin. Table 1 summarizes and compares the available, above-mentioned analytical models.

However, compared with the studies mentioned above of groundwater flow in an aquifer with fully or partially penetrating wells, the research on a particular well structure of a single-well circulation system is quite limited. Only a few studies (Ni et al., 2011; Sorensen & Reffstrup, 1992; Tu et al., 2019) dealing with groundwater flow in an aquifer with a single-well circulation system can be found in the literature. Sorensen and Reffstrup (1992) were the first to present a simplified mathematical model for the analysis of groundwater flow and heat transport for a new type of a groundwater heat pump system, which can extract and inject groundwater with one well borehole. Ni et al. (2011) developed a mathematical model describing groundwater flow induced by a single-well system in a leaky confined aquifer. Based on the superposition principle and by adopting the analytical solution of a partially penetrating well derived by Hantush (1961), Ni et al. (2011) obtained their solution by summing up the analytical solutions for these two partially penetrating wells. Although it is convenient to obtain analytical solutions using the superposition principle, this approach inevitably introduces problems such as complicated well configuration functions. The Hantush

<table>
<thead>
<tr>
<th>Model</th>
<th>Well structure</th>
<th>Well radius</th>
<th>Groundwater flow</th>
<th>Skin effect</th>
<th>Type of aquifer</th>
</tr>
</thead>
<tbody>
<tr>
<td>This study</td>
<td>Two partially penetrating wells in one wellbore</td>
<td>Zero</td>
<td>Darcian flow</td>
<td>No</td>
<td>A confined aquifer with an infinite areal extent</td>
</tr>
<tr>
<td>Theis (1935)</td>
<td>A fully penetrating well</td>
<td>Zero</td>
<td>Darcian flow</td>
<td>No</td>
<td>A confined aquifer with an infinite areal extent</td>
</tr>
<tr>
<td>Chen (1984)</td>
<td>A fully penetrating well</td>
<td>Zero</td>
<td>Darcian flow</td>
<td>No</td>
<td>A finite confined aquifer</td>
</tr>
<tr>
<td>Wen and Zhan (2008)</td>
<td>A fully penetrating well</td>
<td>Zero/Finite</td>
<td>non-Darcian flow</td>
<td>No</td>
<td>A confined aquifer with an infinite areal extent</td>
</tr>
<tr>
<td>Lin et al. (2016)</td>
<td>A fully penetrating well</td>
<td>Finite</td>
<td>Darcian flow</td>
<td>Yes</td>
<td>A two-zone confined aquifer with a finite areal extent</td>
</tr>
<tr>
<td>Hantush (1961)</td>
<td>A partially penetrating well</td>
<td>Zero</td>
<td>Darcian flow</td>
<td>No</td>
<td>A confined aquifer with an infinite areal extent</td>
</tr>
<tr>
<td>Chiu et al. (2007)</td>
<td>A partially penetrating well</td>
<td>Zero</td>
<td>Darcian flow</td>
<td>Yes</td>
<td>A two-zone confined aquifer with a finite areal extent</td>
</tr>
<tr>
<td>Wen et al. (2013)</td>
<td>A partially penetrating well</td>
<td>Zero</td>
<td>non-Darcian flow</td>
<td>No</td>
<td>A confined aquifer with an infinite areal extent</td>
</tr>
<tr>
<td>Feng and Wen (2016)</td>
<td>A partially penetrating well</td>
<td>Finite</td>
<td>non-Darcian flow</td>
<td>Yes</td>
<td>A two-zone confined aquifer with an infinite areal extent</td>
</tr>
</tbody>
</table>
and Jacob well function (Hantush, 1961) makes the analytical solution more complicated and increases its instability and computational cost. Subsequently, Tu et al. (2019) derived a closed-form, steady-state analytical solution to investigate groundwater flow in a confined aquifer by employing the separation variable method and the Laplace transform. However, their solution still includes the complicated Hantush and Jacob well function and the well configuration function. Therefore, to overcome the above-mentioned problems, we aim at rigorously deriving the analytical solution based on the governing equation for a single-well circulation system by employing a combination of the Laplace and Fourier cosine transforms. The main objective of this work is to develop a general analytical mathematical model describing groundwater flow in a confined aquifer for a single-well circulation system with a unique well structure with two partially penetrating wells in one single wellbore. The analytical solution for drawdown in the Laplace domain is obtained by employing the Laplace and Fourier cosine transforms. The effects of the hydraulic conductivity, the length of the sealed section, and other parameters on the drawdown distribution were analyzed, which to our knowledge have not been investigated by other researchers. The steady-state solution in the time domain is also derived. The analytical solution of transient drawdown is used in the sensitivity analysis to investigate the effects of different parameters on drawdown and to identify which of these parameters must be carefully considered when designing a single-well circulation system.

2. Mathematical Model

In order to investigate groundwater flow in a single-well circulation system, a mathematical model was developed, as shown in Figure 2. The assumptions of this mathematical model are as follows: (1) A confined aquifer is assumed to be homogeneous, anisotropic, and infinite in the horizontal direction; (2) underlying and overlying rocks are considered to be impermeable, homogeneous, anisotropic, and of uniform thickness; (3) groundwater flow in a confined aquifer follows Darcy’s law; (4) the storage and release of groundwater due to rising and decreasing hydraulic heads are instantaneous; (5) pumping and injection rates $Q$ (positive for pumping and negative for injection) are constant; and (6) the well radius is infinitesimally small and can be regarded as 0.

2.1. Governing Equation

Base on the above-listed assumptions, the equation describing groundwater flow in a confined aquifer can be written as follows:

$$
K_r \left( \frac{\partial^2 s(r, z, t)}{\partial r^2} + \frac{1}{r} \frac{\partial s(r, z, t)}{\partial r} \right) + K_z \frac{\partial^2 s(r, z, t)}{\partial z^2} = \frac{s}{\partial t},
$$

where $r$ is the radial coordinate (L); $t$ denotes time (T); $s$ is groundwater drawdown (L); $K_r$ and $K_z$ represent the horizontal and vertical hydraulic conductivities, respectively (LT$^{-1}$); and $S$ refers to the specific storage of an aquifer (L$^{-1}$).

2.2. Initial Conditions

Initial drawdown in a confined aquifer at time $t = 0$ is assumed to be constant and equal to 0:

$$
s(r, z, 0) = 0.
$$

2.3. Boundary Conditions

The boundary conditions in infinity in the radial direction and at the top and bottom of a confined aquifer, $r = \infty$, $z = d$, and $z = 0$, respectively, are given as follows:

$$
s(\infty, z, t) = 0
$$

and

$$
\frac{\partial s(r, d, t)}{\partial z} = 0
$$

and
\[
\frac{\partial s(r, 0, t)}{\partial z} = 0,
\]

where \(z\) is the vertical coordinate (L) and \(d\) denotes the thickness of a confined aquifer (L). The boundary conditions at the rim of the wellbore are as follows:

\[
\lim_{r \to 0} r \frac{\partial s}{\partial r} = \begin{cases} 
-\frac{Q}{2\pi Kr d_1} & (0 \leq z \leq d_1) \\
0 & (d_1 \leq z \leq d_1 + d_2) \\
\frac{Q}{2\pi Kr d_3} & (d_1 + d_2 \leq z \leq d)
\end{cases}
\]

where \(d_1, d_2, \) and \(d_3\) are the lengths of pumping, sealed, and injection sections, respectively (L), and \(Q\) represents pumping and injection rates (L³T⁻¹), which are the same and constant in time.

2.4. A General (Transient) Analytical Solution

The mathematical model of a single-well circulation system is fully described using the governing partial differential equation (Equation 1), initial conditions (Equation 2), and boundary conditions (Equations 3–6). This system of equations can be solved by applying the Laplace and Fourier cosine transforms, which convert partial differential equations to ordinary differential equations. A detailed derivation of the analytical solution is given in Appendix A. The analytical solution of drawdown in the Laplace domain is expressed as

\[
s(r, z, p) = \sum_{N=1}^\infty \frac{Q}{NpK_r \pi^2} \left[ \sin\left(\frac{N\pi d_1}{d}\right) + \sin\left(\frac{N\pi(d_1 + d_2)}{d_3}\right) \right] K_0\left(\sqrt{\delta r}\right) \cos\left(\frac{N\pi z}{d}\right),
\]

in which \(\delta = K_d N^2 \pi^2 + \frac{Sp}{K_r} K_0(\cdot)\) denotes zeroth-order modified Bessel functions of the second kind, and \(p\) and \(N (N = 1, 2, 3,\ldots)\) are the Laplace and Fourier transform variables, respectively.

It is currently not possible to obtain the analytical solution of Equation 7 in the real-time domain due to the Bessel function. However, Equation 7 can be numerically inverted to the real-time domain using several numerical inversion methods (e.g., Crump, 1976; De Hoog et al., 1982; Stehfest, 1970a, 1970b). In this study, the Stehfest method is utilized to invert the Laplace domain solution numerically and to calculate drawdown in the real-time domain. The detailed introduction of the Stehfest method can be found in Stehfest (1970a, 1970b).

A MATLAB program has been developed to calculate drawdown in the Laplace domain, which is then inverted numerically using the Stehfest method. This inversion method requires as input the number of terms used in the inversion, \(n\), and the number of terms of the infinite series, \(N\). Numerical tests indicate that the choice of \(n = 20\) and \(N \geq 10\) gives accurate and stable results. To reduce the computational cost, \(n = 20\) and \(N = 10\) are used in all calculations.

2.5. A Steady-State Analytical Solution

When the pumping and injection times are large, namely, \(t \to \infty\), the groundwater flow will reach a quasi-steady state. In this case, drawdown around partially penetrating wells stops changing with time. Therefore, we have

\[
\frac{\partial s(r, z, t)}{\partial t} = 0.
\]

Then, Equation 1 can be reduced to

\[
K_r \left( \frac{\partial^2 s(r, z)}{\partial r^2} + \frac{1}{r} \frac{\partial s(r, z)}{\partial r} \right) + K_d \frac{\partial^2 s(r, z)}{\partial z^2} = 0.
\]
Applying the Fourier cosine transform to the second-order partial derivative with respect to the \( z \) coordinate, the partial differential equation 9 is solved (see Appendix B) to give

\[
s(r, z) = \sum_{N=1}^{\infty} \frac{Q}{NK_r \pi^2} \left[ \frac{\sin(N\pi d_1)}{d_1} + \frac{\sin(N\pi (d_1 + d_2))}{d_3} \right] K_0\left(\sqrt{\delta} r\right) \cos\left(\frac{N\pi z}{d}\right),
\]

where \( \delta = \frac{K_z (N\pi)^2}{K_r} \).

According to Darcy’s Law, steady-state groundwater fluxes can be obtained from the analytical solution for steady-state drawdown (Equation 10) as follows:

\[
\begin{align*}
\nu_r &= K_r \frac{\partial \delta}{\partial r}, \\
\nu_z &= K_r \frac{\partial \delta}{\partial z}.
\end{align*}
\]

According to the properties of the modified Bessel function, the mathematical relation \( \frac{dK_0(x)}{dx} = -K_1(x) \), where \( x \) is the variable of the modified Bessel function. Thus, Equation 11 can be written as

\[
\begin{align*}
\nu_r &= \sum_{N=1}^{\infty} \frac{-Q}{\pi d} \sqrt{\frac{K_z}{K_r}} \left[ \frac{\sin(N\pi d_1)}{d_1} + \frac{\sin(N\pi (d_1 + d_2))}{d_3} \right] K_1\left(\sqrt{\delta} r\right) \cos\left(\frac{N\pi z}{d}\right), \\
\nu_z &= \sum_{N=1}^{\infty} \frac{-QK_z}{\pi dK_r} \left[ \frac{\sin(N\pi d_1)}{d_1} + \frac{\sin(N\pi (d_1 + d_2))}{d_3} \right] K_0\left(\sqrt{\delta} r\right) \sin\left(\frac{N\pi z}{d}\right).
\end{align*}
\]

### 3. Results and Discussion

#### 3.1. Verification of the Analytical Solution

First, the newly derived analytical solution for transient drawdown is verified by comparing its results with the solution of Ni et al. (2011). In their work, a single-well circulation system is regarded as a combination of one partially penetrating well for injection and another one for pumping. The analytical solution for a partially penetrating well was developed by Hantush (1961). Based on the superposition principle, Ni et al. (2011) obtained the analytical solution by summing up the analytical solutions for these two partially penetrating wells in a single-well circulation system. These analytical solutions (i.e., Equation 7 and Ni et al., 2011) for a single-well circulation system are used to calculate transient groundwater drawdown for a system with the following parameters: \( Q = 60 \text{ m}^3/\text{hr} \), \( K_r = 0.1 \text{ m/hr} \), \( K_z = 0.01 \text{ m/hr} \), \( S = 0.0001 \text{ m}^{-1} \), \( d = 40 \text{ m} \), \( d_1 = 15 \text{ m} \), \( d_2 = 10 \text{ m} \), and \( d_3 = 15 \text{ m} \) at \( r = 5 \text{ m} \) and \( z = 15 \text{ m} \). Figure 3 shows that the results of the newly derived analytical solution correspond very well with the solution of Ni et al. (2011).

#### 3.2. Steady-State Drawdown

Figure 4 depicts contours for steady-state drawdown and steady-state flux vectors (Equations 10 and 12, respectively) in a single-well circulation system.
The following parameters (the same as in Figure 3) were used in this scenario: $Q = 60 \text{ m}^3/\text{hr}$, $K_r = 0.1 \text{ m/hr}$, $K_z = 0.01 \text{ m/hr}$, $S = 0.0001 \text{ m}^{-1}$, $d = 40 \text{ m}$, $d_1 = 15 \text{ m}$, $d_2 = 10 \text{ m}$, and $d_3 = 15 \text{ m}$. It can be clearly observed that contours for steady-state drawdown are symmetric around a horizontal midplane of a confined aquifer and vary tremendously with distance from the well axis. The contours of drawdown around the sealed section are dense, indicating that the hydraulic gradient in this area is relatively large. Additionally, due to flow symmetry, absolute values of drawdown are very small around the midplane of a confined aquifer; drawdown is 0 at locations with $z = 20 \text{ m}$ (the middle of a confined aquifer).

### 3.3. Radial Hydraulic Conductivity

Figure 5 shows the effects of different values of the radial hydraulic conductivity ($K_r = 0.01, 0.02, 0.05, \text{ and } 0.1 \text{ m/hr}$) on drawdown with time. Other parameters remain the same as in previous examples; that is, $Q = 60 \text{ m}^3/\text{hr}$, $K_z = 0.01 \text{ m/hr}$, $S = 0.0001 \text{ m}^{-1}$, $d = 40 \text{ m}$, $d_1 = 15 \text{ m}$, $d_2 = 10 \text{ m}$, and $d_3 = 15 \text{ m}$, at $r = 5 \text{ m}$ and $z = 15 \text{ m}$. It can be seen that drawdown increases with the radial hydraulic conductivity ($K_r$) at early times, while it decreases as $K_r$ increases at late times. Larger hydraulic conductivities will cause groundwater to react more quickly to the start of water pumping/injection from/into the aquifer at early times, with the depression cone in the suction zone spreading faster, while groundwater can be replenished in a timely manner. Drawdown is thus larger at early times for larger hydraulic conductivities. In contrast, at late times, an increase in drawdown for larger hydraulic conductivities is not as large as for smaller hydraulic conductivities. Moreover, it is worth noting that drawdown converges toward a steady state more quickly for larger radial hydraulic conductivities, as indicated in Figure 5.

### 3.4. Sealed Section

Figure 6 shows the impact of the length of the sealed section ($d_2$) on groundwater drawdown in a single-well circulation system. The sealed section is located in the middle of the confined aquifer and between the pumping and injection sections of the well (see Figure 2). The same parameters as in the previous section are used to obtain results displayed in Figure 6, that is, $Q = 60 \text{ m}^3/\text{hr}$, $K_z = 0.01 \text{ m/hr}$, $S = 0.0001 \text{ m}^{-1}$, $d = 40 \text{ m}$, $d_1 = 15 \text{ m}$, and $d_3 = 15 \text{ m}$ (i.e., $d_1 = d_3$), at $r = 5 \text{ m}$ and $z = 15 \text{ m}$. Besides, the following lengths of the sealed section are used: $d_2 = 6, 8, 10, 12, \text{ and } 14 \text{ m}$, while $d_1$ and $d_3$ were correspondingly adjusted to keep $d$ constant. One can observe in Figure 6a that shorter lengths of the sealed section result in larger drawdown at early times, while the length of the sealed section has only a little effect on drawdown at late times. This is due to the fact that shorter lengths of the sealed section require longer lengths of the pumping and injection sections, which require smaller pumping and injection rates for a unit well length (note that the total pumping and/or injection rates are constant). Thus, shorter lengths of the sealed section produce larger

![Figure 5](image_url)

*Figure 5.* The effect of the radial hydraulic conductivity ($K_r$) on drawdown with time (for parameters given in the figure).

![Figure 6](image_url)

*Figure 6.* The effect of the length of the sealed section ($d_2$) when $d_1 = d_3$ (a) and various ratios $\kappa = (d_1/d_3)$ for a constant $d_2$ (b) on drawdown with time.
drawdowns at early times. Moreover, drawdowns approach a steady state (i.e., curves are horizontal and parallel) when \( t > 1 \text{ hr} \), which is different from the previous case (the radial hydraulic conductivity). It is worth noting that the effect of further decreasing the length of the sealed section from 8 to 6 m on drawdown becomes negligible. Differences between drawdowns are larger for larger lengths of the sealed section (e.g., from 10 to 14 m) than for smaller lengths of the sealed section.

The effect of the ratio \( \kappa \) of the lengths of the pumping and injection sections \( (\kappa = \frac{d_1}{d_3}, \text{for } d_1 \neq d_3 \text{ and constant } d_2) \) is shown in Figure 6b. As shown in Figure 6b, larger \( \kappa \) ratios result in smaller drawdowns. Drawdowns converge asymptotically toward a steady state for all scenarios. In addition, it is worth noting that changes in drawdown for smaller \( \kappa \) are larger than those for larger \( \kappa \) (from 1.0 to 1.8).

3.5. Aquifer Specific Storage

Figure 7 demonstrates how drawdown changes with time for different values of the aquifer specific storage \( (S = 0.0001, 0.0002, 0.0005, \text{ and } 0.001 \text{ m}^{-1}) \) and for the same other conditions as before, that is, \( Q = 60 \text{ m}^3/\text{hr}, K_r = 0.1 \text{ m/hr}, K_z = 0.01 \text{ m/hr}, d = 40 \text{ m}, d_1 = 15 \text{ m}, d_2 = 10 \text{ m}, \text{and } d_3 = 15 \text{ m}, \text{ at } r = 5 \text{ m} \) and \( z = 15 \text{ m} \). It can be observed that drawdown decreases with an increase in the specific storage \( S \) at early times. In contrast, all drawdown curves for different values of the specific storage converge asymptotically to the same value at late times. When other conditions remain the same, an aquifer with large specific storage releases more water than an aquifer with small specific storage. Therefore, as shown in Figure 7, drawdown decreases with the specific storage at early times. Since the process of water release from the aquifer storage is almost complete at late times, all drawdown curves eventually approach the same asymptotic value after about 10 hr of operation and remain stabilized at late times. As a result, the impact of the specific storage on drawdown is insignificant at late times.

3.6. Pumping Time

It is also interesting to analyze how drawdown varies with distance for different pumping times \( (t = 0.1, 1, \text{ and } 10 \text{ hr}, \text{ and steady state}) \). The other parameters in this scenario are again the same as before; that is, \( Q = 60 \text{ m}^3/\text{hr}, K_r = 0.1 \text{ m/hr}, K_z = 0.01 \text{ m/hr}, S = 0.0001 \text{ m}^{-1}, d = 40 \text{ m}, d_1 = 15 \text{ m}, d_2 = 10 \text{ m}, \text{ and } d_3 = 15 \text{ m}, \text{ at } r = 5 \text{ m} \) and \( z = 15 \text{ m} \). The analytical solution Equation 10 for the steady-state drawdown (see section 2.4) is utilized as a reference in this section. Figure 8 shows that drawdown changes only slightly with time for relatively small distances from the well, indicating that groundwater flow quickly reaches a quasi-steady state in the area close to partially penetrating wells. Additionally, Figure 8 also shows that the curves for \( t = 10 \text{ hr} \) and the steady state almost coincide, indicating that the entire flow field in the single-well circulation system reaches steady state relatively quickly.

3.7. Sensitivity Analysis

To estimate the effects of various parameters on drawdown, the sensitivity analysis has been carried out for the specific storage \( S \), the length of the sealed section \( d_3 \), and the radial and vertical hydraulic conductivities, \( K_r \) and \( K_z \), respectively. Using the definition of the normalized parameter sensitivity proposed by Kabala (2001), one can obtain

\[
Y_{m,n} = \frac{\Delta X_m}{\Delta R_n} \quad (11)
\]

where \( Y_{m,n} \) denotes the normalized sensitivity of the \( m \)th parameter \( R_n \) at the \( n \)th time and \( X_m \) is a function of \( R_n \) and \( n \). A finite-difference equation proposed by Yeh (1986) (also used by Simunek & van
Genuchten, 1996) can be used to approximate the partial derivative in Equation 11:

$$ \frac{\partial X_m}{\partial R_n} = \frac{X_m(R_n + \Delta R_n) - X_m(R_n)}{\Delta R_n} $$

(12)

The small increment $\Delta R_n$ of $0.01 \times R_n$ is used in this work, similarly as in Yeh (1986), Simunek and van Genuchten (1996), Huang and Yeh (2007), and Wen et al. (2013), with the latter two carrying out the sensitivity analysis of parameters for non-Darcian flow problems in a confined aquifer. Other parameters are as before: $Q = 60 \text{ m}^3/\text{hr}$, $K_r = 0.1 \text{ m/hr}$, $K_z = 0.01 \text{ m/hr}$, $S = 0.0001 \text{ m}^{-1}$, $d = 40 \text{ m}$, $d_1 = 15 \text{ m}$, $d_2 = 10 \text{ m}$, and $d_3 = 15 \text{ m}$, at $r = 5 \text{ m}$ and $z = 15 \text{ m}$. Figure 8 shows the normalized sensitivities of drawdown with respect to the change of selected parameters. The sensitivity analysis indicates that these parameters do not have any impact on drawdown at early times and that each parameter has its own influence period on drawdown. It also indicates that drawdown is very sensitive to the change in the radial hydraulic conductivity. It is worth noting that the radial hydraulic conductivity $K_r$ has a negative influence on drawdown in the period of $3 \times 10^{-3} \text{ hr} < t < 2 \times 10^{-1} \text{ hr}$. The normalized sensitivity then starts increasing until about 4 hr when it stabilizes. The sensitivity of drawdown with respect to the lengths of the sealed section $d_2$ starts at about $3 \times 10^{-3} \text{ hr}$, reaches the highest value at about 0.5 hr, and then remains stable. The sensitivity of drawdown associated with the specific storage $S$ starts increasing at about $3 \times 10^{-3} \text{ hr}$, reaches the peak value at about 0.1 hr, and then gradually decreases until about 15 hr. Finally, the influence of the vertical hydraulic conductivity on drawdown is small and negligible before 0.1 hr, then keeps increasing until about 10 hr, and then remains constant.

In practical engineering applications, the design of a single-well circulation groundwater heat pump needs to consider many factors, such as local hydrogeological conditions, thermal geological conditions, and the configuration of the well structure, in order to optimize the efficiency of the system and minimize its costs. As indicated by the sensitivity analysis in Figure 9, it is necessary to investigate hydrogeological conditions using pumping tests, in situ permeability tests, and geophysical surveys to obtain data related to hydraulic conductivities. The configuration of the well structure in real engineering applications then needs to be adjusted according to aquifer properties. As shown in Figure 6, shorter lengths of the sealed section result in larger drawdown, which indicates that the thermal breakthrough will occur sooner and the system will be less efficient. Although longer lengths of the sealed section could delay the thermal breakthrough, the cost of drilling the well and the electricity consumption for pumping and injection will increase. Therefore, factors such as the efficiency of the system, initial costs, and energy consumption should be comprehensively considered when designing a proper length of the sealed section for a single-well circulation groundwater heat pump system.

4. Conclusions

A general analytical mathematical model was developed in this study to investigate the characteristics of groundwater flow in a single-well circulation system. The governing partial differential equation is solved simultaneously with initial and boundary conditions by applying the Laplace and Fourier cosine transforms. The analytical solution for transient drawdown in the Laplace domain is obtained and then inverted numerically using the Stehfest method. The steady-state solution for drawdown in the time domain for a single-well circulation system is also developed. Additionally, the effects of different parameters on drawdown are also analyzed. The main research conclusions can be drawn as follows:

1. The contours for steady-state drawdown are symmetric around the horizontal midline of the aquifer and vary tremendously with distance from the well axis (in the case of $d_1 = d_3$). The drawdown contours around the sealed section are dense, indicating that the hydraulic gradient in this area is relatively large.

Figure 9. The sensitivity analysis of drawdown to selected parameters (the specific storage $S$, the length of the sealed section $d_2$, and the radial and vertical hydraulic conductivities, $K_r$ and $K_z$, respectively).
2. The length of the sealed section $d_2$ has a significant influence on drawdown in the case of $d_1 = d_3$. In the case of $d_1 \neq d_3$ and a constant length of the sealed section $d_2$, larger ratios of $\kappa (\kappa = \frac{d_1}{d_3})$ result in smaller drawdowns, while changes in drawdown become insignificant for further increases in $\kappa$ values.

3. The spatial distribution of drawdown in a single-well circulation system is sensitive to the radial hydraulic conductivity $K_r$ and the length of the sealed section $d_2$. Each parameter has its own influence period on drawdown.

4. Drawdown is not sensitive to parameters $S, d_2, K_r$, and $K_z$ at early times, while it is very sensitive to these parameters at late times, particularly to the radial hydraulic conductivity, $K_r$.

Appendix A: Derivation of the Analytical Solution of the Mathematical Model

The governing equation in the Laplace domain is obtained by using the Laplace transform with respect to time $t$:

$$
\frac{\partial^2 \tilde{z}(r, z, p)}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{z}(r, z, p)}{\partial r} + \frac{K_z}{K_r} \frac{\partial^2 \tilde{z}(r, z, p)}{\partial z^2} = \frac{S}{K_r} \tilde{s}(r, z, p),
$$

(A1)

where $\tilde{z}$ is the Laplace transform of drawdown and $p$ is the Laplace transform variable. The Fourier cosine transform is then applied to the term in Equation A1 with the second-order partial derivative with respect to the $z$ coordinate:

$$
F_z \left[ \frac{\partial^2 \tilde{z}(r, z, p)}{\partial z^2} \right] = \int_0^d \frac{\partial^2 \tilde{z}(r, z, p)}{\partial z^2} \cos \left( \frac{N \pi z}{d} \right) dz = \frac{N^2 \pi^2}{d^2} \tilde{s}(r, N, p).
$$

(A2)

Equation A1 is then rewritten by substituting Equation A2 as follows:

$$
\frac{d^2 \tilde{z}(r, N, p)}{dr^2} + \frac{1}{r} \frac{d \tilde{z}(r, N, p)}{dr} + \frac{K_z}{K_r} \left( \frac{-N^2 \pi^2 \delta}{d^2} \right) \tilde{s}(r, N, p) = \frac{Sp}{K_r} \tilde{s}(r, N, p).
$$

(A3)

in which $\tilde{s}$ denotes the Fourier cosine transform of drawdown and $N (N = 1, 2, 3, \ldots)$ is the Fourier transform variable. After simplifying Equation A3, one obtains

$$
\frac{d^2 \tilde{z}(r, N, p)}{dr^2} + \frac{1}{r} \frac{d \tilde{z}(r, N, p)}{dr} - \left( \frac{K_z N^2 \pi^2}{K_r d^2} + \frac{Sp}{K_r} \right) \tilde{s}(r, N, p) = 0.
$$

(A4)

Equation A4 can be rewritten as follows by defining $\delta = \frac{K_z N^2 \pi^2}{K_r d^2} + \frac{Sp}{K_r}$

$$
\frac{d^2 \tilde{z}(r, N, p)}{dr^2} + \frac{1}{r} \frac{d \tilde{z}(r, N, p)}{dr} - \delta \tilde{s}(r, N, p) = 0.
$$

(A5)

Equation A5 is a linear second-order differential equation, which has the general solution as follows:

$$
\tilde{s}(r, N, p) = C_1 I_0 \left( \sqrt{\delta} r \right) + C_2 K_0 \left( \sqrt{\delta} r \right),
$$

(A6)

where $C_1$ and $C_2$ are integration constants, which can be determined using boundary conditions, and $I_0(\chi)$ and $K_0(\chi)$ denote the first and second kind modified Bessel functions of zeroth order with an argument $\chi$, respectively.

Applying the Laplace transform to Equations 3–5, the boundary conditions in the Laplace domain can be given as follows, respectively:

$$
\frac{\partial \tilde{s}(r, 0, p)}{\partial z} = 0,
$$

(A7)

$$
\tilde{s}(\infty, z, p) = 0.
$$

(A8)
\[
\frac{\partial \xi(r, D, p)}{\partial z} = 0. \tag{A9}
\]

Using the Fourier cosine transform to the \( z \) coordinate in Equations A7–A9, we can obtain
\[
\xi(\infty, N, p) = 0. \tag{A10}
\]

Taking into account the property of modified Bessel functions of the first kind, substituting Equation A10 into Equation A6, and having \( C_1 = 0 \), Equation A6 then can be rewritten as
\[
\xi(r, N, p) = C_2 K_0 \left( \sqrt{\frac{\delta p}{r}} \right). \tag{A11}
\]

Applying the Laplace and Fourier cosine transforms to Equation 6, one can obtain, respectively,
\[
\lim_{r \to 0} r \frac{\partial \xi(r, z, p)}{\partial r} = \begin{cases} \frac{-Q}{2\pi K_r d_i p} & (0 \leq z \leq d_1) \\ 0 & (d_1 \leq z \leq d_1 + d_2) \\ \frac{Q}{2\pi K_r d_i p} & (d_1 + d_2 \leq z \leq d) \end{cases} \tag{A12}
\]

and
\[
\lim_{r \to 0} \frac{d\xi(r, N, p)}{dr} = -\frac{dQ}{2NpK_r d_i \pi^2} \left[ \frac{\sin \left( \frac{N\pi d_i}{d} \right)}{d_1} + \frac{\sin \left( \frac{N\pi(d_1 + d_2)}{d} \right)}{d} \right]. \tag{A13}
\]

From Equations A11 and A13, we can obtain the integration constant \( C_2 \):
\[
C_2 = \frac{dQ}{2NpK_r \pi^2} \left[ \frac{\sin \left( \frac{N\pi d_i}{d} \right)}{d_1} + \frac{\sin \left( \frac{N\pi(d_1 + d_2)}{d} \right)}{d} \right]. \tag{A14}
\]

Substituting Equation A14 into Equation A11, we get
\[
\xi(r, N, p) = \frac{dQ}{2NpK_r \pi^2} \left[ \frac{\sin \left( \frac{N\pi d_i}{d} \right)}{d_1} + \frac{\sin \left( \frac{N\pi(d_1 + d_2)}{d} \right)}{d} \right] K_0 \left( \sqrt{\frac{\delta p}{r}} \right). \tag{A15}
\]

When \( N = 0 \), the term \( \frac{K_z N^2 \pi^2}{K_r} \frac{d^2}{d^2} + \frac{Sp}{K_r} \) in Equation A4 becomes \( \frac{Sp}{K_r} \). Here, we define \( \omega = \frac{Sp}{K_r} \). Thus, the governing equation for this case can be written as
\[
\frac{d^2 \xi(r, 0, p)}{dr^2} + \frac{1}{r} \frac{d\xi(r, 0, p)}{dr} - \omega \xi(r, 0, p) = 0. \tag{A16}
\]

A similar approach is applied to solve Equation A16 and boundary conditions. Finally, the general solution can be obtained as follows:
\[
\xi(r, 0, p) = C_2' K_0 \left( \sqrt{\omega r} \right). \tag{A17}
\]

Applying the Fourier cosine transform to Equation A12, one has
\[
\lim_{r \to 0} \frac{d\xi(r, 0, p)}{dr} = \int_0^{d_i} \frac{-Q}{2\pi K_r d_i p} \cos \left( \frac{N\pi z}{d} \right) dz \bigg|_{N=0} + \int_{d_i+d_2}^{d} \frac{Q}{2\pi K_r d_i p} \cos \left( \frac{N\pi z}{d} \right) dz \bigg|_{N=0} = 0. \tag{A18}
\]

Based on Equation A17, one can obtain
lim_{r\to0} \frac{d\tilde{s}(r, 0, p)}{dr} = -\lim_{r\to0} \left[C'_2 \sqrt{\omega r} K_1(\sqrt{\omega r})\right] = -C'_2. \quad (A19)

Then, combining Equations A18 and A19, the integration constant $C'_2$ in the case of $N=0$ can be obtained as

$$C'_2 = 0. \quad (A20)$$

Then, substituting Equation A20 to Equation A17, one has

$$\tilde{s}(r, 0, p) = 0. \quad (A21)$$

Using the inverse Fourier cosine transform to Equation A15, one can obtain

$$\tilde{s}(r, z, p) = \frac{\tilde{s}(r, 0, p)}{d} + \frac{2}{d} \sum_{N=1}^{\infty} \tilde{s}(r, N, p) \cos\left(\frac{N\pi z}{d}\right). \quad (A22)$$

Finally, by substituting Equations A15 and A21 to Equation A22, we can obtain an analytical solution in the Laplace domain.

**Appendix B: Derivation of the Steady-State Solution**

Applying the Fourier cosine transform to the term of the second-order partial derivative of the $z$ coordinate in Equation 9, one can obtain

$$F_c \left[\tilde{s}(r, z, p)\right] = \int_0^\infty \tilde{s}(r, z) \cos \left(\frac{N\pi z}{d}\right) dz = -\frac{N^2\pi^2}{d^2} \tilde{s}(r, N). \quad (B1)$$

Substituting Equation B1 to Equation 9, one gets

$$\frac{d^2\tilde{s}(r, N)}{dr^2} + \frac{1}{r} \frac{d\tilde{s}(r, N)}{dr} + \frac{K_z}{K_r} \left(-\frac{N^2\pi^2}{d^2}\right) \tilde{s}(r, N) = 0. \quad (B2)$$

By defining $\delta = \frac{K_z}{K_r} \frac{N^2\pi^2}{d^2}$, Equation B2 can be simplified as follows:

$$\frac{d^2\tilde{s}(r, N)}{dr^2} + \frac{1}{r} \frac{d\tilde{s}(r, N)}{dr} - \delta \tilde{s}(r, N) = 0. \quad (B3)$$

The general solution of Equation B3 can be given as follows:

$$\tilde{s}(r, N) = C_3 I_0\left(\sqrt{\delta} r\right) + C_4 K_0\left(\sqrt{\delta} r\right). \quad (B4)$$

where $C_3$ and $C_4$ are integration constants. Using the boundary condition of Equation 3 and considering the property of the first kind of modified Bessel functions, one can have $C_3 = 0$, then Equation B4 becomes

$$\tilde{s}(r, N) = C_4 K_0\left(\sqrt{\delta} r\right). \quad (B5)$$

Applying the Fourier cosine transform to Equation 6, one gets

$$\lim_{r\to0} \frac{d\tilde{s}(r, N)}{dr} = \frac{-dQ}{2NK_d \pi^2} \sin\left(\frac{N\pi d_1}{d}\right) - \frac{dQ}{2NK_d \pi^2} \left[\sin\left(\frac{N\pi (d_1 + d_2)}{d}\right)\right]. \quad (B6)$$

By integrating Equations B5 and B6, the constant $C_4$ can be obtained:
\[ C_4 = \frac{dQ}{2NK_r r^2} \left[ \sin \left( \frac{N\pi d_1}{d} \right) \frac{d_1}{d_1} + \sin \left( \frac{N\pi (d_1 + d_2)}{d} \right) \frac{d_2}{d_3} \right]. \quad (B7) \]

Substituting Equation B7 into Equation B5, one gets

\[ \tilde{s}(r, N) = \frac{dQ}{2NK_r r^2} \left[ \sin \left( \frac{N\pi d_1}{d} \right) \frac{d_1}{d_1} + \sin \left( \frac{N\pi (d_1 + d_2)}{d} \right) \frac{d_2}{d_3} \right] K_0 \left( \sqrt{N} \tilde{r} \right). \quad (B8) \]

Using the inverse Fourier cosine transform to Equation B8, one can obtain

\[ s(r, z) = \frac{1}{d} \tilde{s}(r, 0) + \frac{2}{d} \sum_{n=1}^{\infty} \tilde{s}(r, N) \cos \left( \frac{N\pi z}{d} \right) \quad (B9) \]

Finally, substituting Equation B8 to Equation B9, an analytical solution for steady-state conditions can be obtained (Equation 10).

**Data Availability Statement**

There are no data-sharing issues because any data used in this manuscript have been provided. Computer codes used in this manuscript are available upon request from the corresponding author.

**References**


Theis, C. V. (1935). The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using groundwater storage. Eos, Transactions American Geophysical Union, 16(2), 519–524.


